In other words the considered flow is stable, if inequalities (1.1) and the condition derived in Sects. 2–4 which relates to the particular flow are satisfied at the shocks.

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## ON THE PROBLEM OF OPTIMIZATION OF THE SHAPE OF A BODY IN A VISCOUS FLUID

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We obtain necessary conditions for minimum drag of a body in a viscous fluid when the flow is described either by the exact Navier-Stokes equations or by the approximate Oseen equations. We study some of the characteristics of optimal bodies. The problem of optimizing the shape of a body in the flow of a viscous fluid was considered previously in [1] in the Stokes approximation, wherein necessary conditions were derived which the shape of a body of minimum drag must satisfy; some qualitative characteristics of optimal shapes were also investigated.

**1.** The stationary flow of a viscous incompressible fluid over a body $S$ is described by the Navier-Stokes equations and the no-slip boundary conditions on the body surface. For convenience in our transformations we consider, in the sequel, a finite volume of fluid $\Omega$, bounded in its interior by the surface of a body $S$ and, on the outside, by a surface $\Sigma$ on which the velocity vector $\mathbf{u}$ is specified. For the case in which an unbounded mass of fluid flows over the body the minimum distance from the body surface $S$ to the surface $\Sigma$ must tend to infinity.

We consider the following variational problem: find, among bodies of specified volume $Q$, a body $S$ for which the energy dissipation rate $G$ is a minimum. Here we assume that the velocity distribution on the surface $\Sigma$ does not depend on the shape of the body. We note that if the flow on the surface $\Sigma$ is a translational flow $\mathbf{u} = \mathrm{const}$ then the magnitude of the drag is equal to $G / u$ and the problem of the minimum energy dissipation rate becomes equivalent to the minimum drag problem.

In dimensionless variables the equations of motion of the fluid, the boundary conditions, and the minimizing functional have the form

$$\triangle \mathbf{v} - \nabla p - R(\mathbf{v}\nabla)\mathbf{v} = 0, \quad \nabla \mathbf{v} = 0, \quad \mathbf{v}|_S = 0, \quad \mathbf{v}|_\Sigma = \mathbf{u} \tag{1.1}$$

$$G(S) = \int_\Omega \frac{1}{2} \sum_{i,j=1}^{3} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 d\Omega \tag{1.2}$$

**2.** We obtain a necessary condition for the functional (1.2) to be a minimum, subject to the differential constraints (1.1). Let the surface of the body $S_0$ be specified in the parametric form $x_i = x_{0i}(q, r)$, where $q$ and $r$ are parameters. Consider the family of bodies $S_\varepsilon$, defined by the equations

$$x_i = x_{0i}(q, r) + \varepsilon n_i f(q, r), \quad 0 < \varepsilon \ll 1$$

where $n_i$ are the components of the unit exterior normal vector to the surface $S_0$, and $f(q, r)$ is a fixed function. Let $\mathbf{v}_\varepsilon$ and $p_\varepsilon$ denote the solution of the boundary value problem (1.1) with boundary conditions on the surface $S_\varepsilon$. We write the quantities $\mathbf{v}_\varepsilon$ and $p_\varepsilon$ in the form

$$\mathbf{v}_\varepsilon = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + o(\varepsilon), \quad p_\varepsilon = p_0 + \varepsilon p_1 + o(\varepsilon) \tag{2.1}$$

The functions $\mathbf{v}_0$ and $p_0$ satisfy the boundary value problem (1.1) for the body $S_0$.

We take the boundary conditions for the velocity on the surface $S_0$ with due regard to boundary conditions for $\mathbf{v}_0$. At the outer boundary $\Sigma$ of the fluid we assume the boundary conditions to be independent of the shape of the body $S$. Thus, for the functions $\mathbf{v}_1$ and $p_1$ according to (1.1), (2.1), we have the boundary value problem

$$\triangle \mathbf{v}_1 - \nabla p_1 - R[(\mathbf{v}_0\nabla)\mathbf{v}_1 + (\mathbf{v}_1\nabla)\mathbf{v}_0] = 0, \quad \nabla \mathbf{v}_1 = 0 \tag{2.2}$$

$$\mathbf{v}_1|_S = -f\frac{\partial \mathbf{v}_0}{\partial n}, \qquad \mathbf{v}_1|_\Sigma = 0$$

In the notation adopted here, the functional (1.2), calculated for the body $S_\varepsilon$, has the form

$$G(S_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} \sum_{i,j=1}^{3} \left( \frac{\partial v_{\varepsilon i}}{\partial x_j} + \frac{\partial v_{\varepsilon j}}{\partial x_i} \right)^2 d\Omega = G(S_0) + \tag{2.3}$$

$$\varepsilon \int_{\Omega_0} \sum_{i,j=1}^{3} \left( \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{0j}}{\partial x_i} \right) \left( \frac{\partial v_{1i}}{\partial x_j} + \frac{\partial v_{1j}}{\partial x_i} \right) d\Omega - \frac{1}{2} \varepsilon \int_{S_0} f \sum_{i,j=1}^{3} \left( \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{0j}}{\partial x_i} \right)^2 dS$$

Transforming the first integral in Eq. (2.3) through an integration by parts and taking into account the boundary conditions for the functions $\mathbf{v}_0$ and $\mathbf{v}_1$, we obtain

$$G(S_\varepsilon) = G(S_0) + \varepsilon \int_{S_0} f \left| \frac{\partial \mathbf{v}_0}{\partial n} \right|^2 dS - 2\varepsilon \int_{\Omega_0} \mathbf{v}_1 \triangle \mathbf{v}_0 d\Omega + o(\varepsilon) \tag{2.4}$$

Let $p^*$ and $\mathbf{v}^*$ be scalar and vector functions defined in $\Omega_0$, and assume that the function $\mathbf{v}^*$ satisfies the zero boundary conditions $\mathbf{v}^*|_\Sigma = \mathbf{v}^*|_S = 0$. We multiply the first of the equations (2.2) scalarly by $\mathbf{v}^*$ and the continuity equation by $p^*$. Adding the resulting expressions and integrating over the domain $\Omega_0$, we have

$$0 = \int_{\Omega_0} \{v_i^* [\triangle v_{1i} - \nabla_i p - R(v_{0j}\nabla_j v_{1i} + v_{1j}\nabla_j v_{0i}) + p^* \nabla_i v_{1i}\} d\Omega$$

Integrating this expression by parts twice, and using the boundary conditions for the functions $v_{1i}$ and $v_i^*$, we obtain (2.5)

$$0 = \int_{S_0} v_{1i} \frac{\partial v_i^*}{\partial n} dS + \int_{\Omega_0} \{v_{1i} [\triangle v_i^* - \nabla_i p^* + R(v_{0j}\nabla_j v_i^* - v_j^* \nabla_i v_{0j})] + p_1 \nabla_j v_j^*\} d\Omega$$

We now determine the functions $v^*$ and $p^*$ as the solution of the boundary value problem

$$\Delta v_i^* - \nabla_i p^* + R(v_{0j}\nabla_j v_i^* - v_j^* \nabla_i v_{0j}) = 2\Delta v_{0i} \qquad (2.6)$$

$$\nabla_i v_i^* = 0, \qquad v_i^*|_\Sigma = v_i^*|_S = 0 \qquad (2.7)$$

Then the expression (2.5) mav be rewritten in the form

$$0 = \int_{S_0} v_{1i} \frac{\partial v_i^*}{\partial n} dS + 2 \int_{\Omega_0} v_{1i} \triangle v_{0i} d\Omega \qquad (2.8)$$

Substituting (2.8) into Eq. (2.4) and taking into account the boundary conditions for the $v_{1i}$, we obtain

$$G(S_\varepsilon) = G(S_0) + \varepsilon \int_{S_0} f\left(\frac{\partial \mathbf{v}_0}{\partial n} \frac{\partial}{\partial n}[\mathbf{v}_0 - \mathbf{v}^*]\right) dS$$

Since the minimum of the functional $G(S)$ is sought in the class of bodies of unit volume, the function $f$ defining the variation of the boundary $S_0$, must satisfy the condition

$$\int_{S_0} f dS = 0 \qquad (2.9)$$

However, the condition that the first variation of the functional $G(S)$ vanish leads to the equation

$$\int_{S_0} f\left(\frac{\partial \mathbf{v}_0}{\partial n} \cdot \frac{\partial}{\partial n}[\mathbf{v}_0 - \mathbf{v}^*]\right) dS = 0 \qquad (2.10)$$

It is evident from a comparison of the expressions (2.9) and (2.10) that the optimal body must satisfy the condition

$$\left(\frac{\partial \mathbf{v}_0}{\partial n} \cdot \frac{\partial}{\partial n}[\mathbf{v}_0 - \mathbf{v}^*]\right)\Big|_{S_0} = \text{const} \qquad (2.11)$$

In other words, the equality (2.11) is a necessary condition for an extremum of the functional $G(S)$ on the class of bodies of constant volume.

Similarly, we can obtain a necessary condition for an extremum of the functional $G(S)$, subject to other isoperimetric conditions. In particular, if we seek the body of minimum drag among bodies with a specified surface area, we find that the variation of the boundary must satisfy the condition

$$\int_{S_0} H f dS = 0$$

where $H$ is the mean curvature of the body surface calculated at an appropriate point. Therefore, the necessary condition for an extremum in this case has the form

$$\frac{1}{H}\left(\frac{\partial \mathbf{v}_0}{\partial n} \cdot \frac{\partial}{\partial n}[\mathbf{v}_0 - \mathbf{v}^*]\right)\Big|_{S_0} = \text{const}$$

3. In the case $R \to 0$, corresponding to the Stokes approximation, Eqs. (1.1) and (2.6) assume the form

$$\triangle \mathbf{v} = \nabla p, \qquad \triangle \mathbf{v}^* = \nabla p^* + 2\triangle \mathbf{v}$$

After substituting the first equation into the second, we obtain

$$\triangle \mathbf{v}^* = \nabla (p^* + 2p) \tag{3.1}$$

However, the boundary value problem (2.7), (3.1) has only the trivial solution $\mathbf{v}^* = 0$, $p^* + 2p = \text{const}$. Therefore, in this case, the necessary condition (2.11) for minimum drag assumes the form

$$\left| \frac{\partial \mathbf{v}}{\partial n} \right|_S^2 = \text{const}$$

which corresponds to the results given in [1].

In giving an approximate description of the motion of a viscous incompressible fluid in the case of a specified uniform translational flow $\mathbf{u} = \text{const}$ at infinity, use is often made of the Oseen equations, the form of which is

$$R(\mathbf{u}\nabla)\mathbf{v} = -\nabla p + \triangle \mathbf{v}, \qquad \nabla \mathbf{v} = 0 \tag{3.2}$$

To obtain a necessary condition for a minimum of the functional (1.2), subject to the differential constraints (3.2), we can follow the same procedure as in the case when the velocity field is determined from the Navier-Stokes equations. We give the final result, omitting the intermediate details. The necessary condition of minimum drag for bodies of constant volume has the form

$$\left( \frac{\partial \mathbf{v}}{\partial n} \cdot \frac{\partial}{\partial n} [\mathbf{v} - \mathbf{v}^*] \right) = \text{const}$$

where the function $\mathbf{v}^*$ is a solution of the boundary value problem

$$\triangle \mathbf{v}^* - \nabla p^* + R(\mathbf{u}\nabla)\mathbf{v}^* = 2\triangle \mathbf{v}, \quad \nabla \mathbf{v}^* = 0, \quad \mathbf{v}^*|_S = \mathbf{v}^*|_\infty = 0 \tag{3.3}$$

Subtracting the first of the equations (3.2) from Eq. (3.3) and introducing the notation $\mathbf{u}' = -\mathbf{u}$, $\mathbf{v}' = \mathbf{v}^* - \mathbf{v}$, $p' = p^* + p$, we obtain

$$R(\mathbf{u}'\nabla)\mathbf{v}' = -\nabla p' + \triangle \mathbf{v}', \quad \nabla \mathbf{v}' = 0, \quad \mathbf{v}'|_\infty = \mathbf{u}', \quad \mathbf{v}'|_S = 0$$

In other words, the function $\mathbf{v}'$ is the solution of the system of Oseen equations in the case in which the translational flow $\mathbf{u}' = -\mathbf{u}$ is specified at infinity. The mechanical significance of the function $\mathbf{v}^*$ is then evident; it is simply the sum of the velocities $\mathbf{v}$ and $\mathbf{v}'$ for translational flows over the body having velocities at infinity of $\mathbf{u}$ and $-\mathbf{u}$ respectively.

4. Consider the plane-parallel flow of a fluid. It follows from the continuity equations in this case that the vector fields $\mathbf{v}$ and $\mathbf{v}^*$ admit stream functions $\psi$ and $\psi^*$

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}, \quad v_x^* = \frac{\partial \psi^*}{\partial y}, \quad v_y^* = -\frac{\partial \psi^*}{\partial x}$$

Here, $x$ and $y$ are Cartesian coordinates in the plane of flow. The equations for the functions $\psi$ and $\psi^*$ are readily found to be

$$R\left( \frac{\partial \psi}{\partial y} \frac{\partial \triangle \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \triangle \psi}{\partial y} \right) = \triangle^2 \psi \tag{4.1}$$

$$\triangle^2 \psi^* - 2\triangle^2 \psi = R\Big[ \frac{\partial \psi}{\partial x} \frac{\partial \triangle \psi^*}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \triangle \psi^*}{\partial x} +$$

$$2\frac{\partial^2\psi}{\partial x\partial y}\left(\frac{\partial^2\psi^*}{\partial y^2}-\frac{\partial^2\psi^*}{\partial x^2}\right)-2\frac{\partial^2\psi^*}{\partial x\partial y}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)\Big]$$

The boundary conditions for the functions $\psi$ and $\psi^*$, and the optimality condition, may be written in the form

$$\psi|_S=\frac{\partial\psi}{\partial n}\Big|_S=0,\quad \frac{\partial\psi}{\partial x}\Big|_\Sigma=-u_y,\quad \frac{\partial\psi}{\partial y}\Big|_\Sigma=u_x \tag{4.2}$$

$$\psi^*|_S=\frac{\partial\psi^*}{\partial n}\Big|_S=\psi^*|_\Sigma=\frac{\partial\psi^*}{\partial n}\Big|_\Sigma=0,\quad \triangle\psi(\triangle\psi-\triangle\psi^*)|_S=\text{const} \tag{4.3}$$

We consider now the asymptotic behavior of the solution of the boundary value problem (4.1)–(4.3) close to the stagnation point (branching point of the streamlines). In the $xy$-plane we take a $\rho$, $\theta$ polar coordinate system, with the pole at the stagnation point and the $\theta=0$ axis directed along the streamline $\psi=0$. We expand the functions $\psi$ and $\psi^*$ in powers of $\rho$

$$\psi=\rho^n f(\theta)+o(\rho^n),\quad \psi^*=\rho^m g(\theta)+o(\rho^m) \tag{4.4}$$

If the distribution of velocities $u$ on the surface $\Sigma$ is symmetric with respect to the $\theta=0$ axis, and if the optimal body is unique, then the body must also be symmetric with respect to this axis. We assume, therefore, that the optimal body is symmetric with respect to the $\theta=0$ axis. In particular, it follows from this that the functions $\psi$ and $\psi^*$ must be odd functions of $\theta$.

Let the surface of the optimal body be given by the equation

$$\theta=\pm\theta_1+O(\rho)$$

where $\theta_1=\text{const}$. Then the boundary conditions for the functions $\psi$ and $\psi^*$ can be written in the form

$$\psi|_S=\rho^n f(\pm\theta_1)=0,\quad \frac{\partial\psi}{\partial n}\Big|_S=\rho^{n-1}f'(\pm\theta_1)=0 \tag{4.5}$$

$$\psi^*|_S=\rho^m g(\pm\theta_1)=0,\quad \frac{\partial\psi^*}{\partial n}\Big|_S=\rho^{m-1}g'(\pm\theta_1)=0$$

Substituting the expressions (4.4) into Eqs. (4.2) and (4.3), we obtain

$$N_n(f)+\rho^n M_n(f)=0,\ N_m(g)+\rho^n L_{mn}(g,f)=2\rho^{n-m}N_n(f) \tag{4.6}$$
$$N_n(f)=(f''+n^2f)''+(n-2)^2(f''+n^2f)$$

Here $N_n$, $(f)$, $M_n(f)$ and $L_{mn}(g,f)$ are certain differential expressions independent of $\rho$ ; the function $\psi$ vanishes at the stagnation point, so that we can assume $n>0$ and discard the second terms on the left-hand sides of Eqs. (4.6). We have

$$N_n(f)=0,\ N_m(g)=2\rho^{n-m}N_n(f)$$

We now substitute the expansions (4.4) into the optimality condition (4.3)

$$[\rho^{2n-4}(f'')^2+\rho^{m+n-4}f''q'']_{\theta=\pm\theta_1}=\text{const}$$

It follows from this formula that the exponents $m$ and $n$ must satisfy one of the conditions: either $n>2$ and $m+n=4$, or $n=2$ and $m+n>4$. Consider the first case. The solution of the system of equations (4.6) then has the form

$$f=\alpha\sin(2+\xi)\theta+\beta\sin\xi\theta \tag{4.7}$$
$$g=\gamma\sin(2-\xi)\theta+\delta\sin\xi\theta$$

where $\xi = 2 - m = n - 2$, and $\alpha$, $\beta$, $\gamma$ and $\delta$ are arbitrary constants. Next, substituting the expressions (4.7) into the boundary conditions (4.5), we obtain a system of equations for $\xi$, $\theta_1$, $\alpha$, $\beta$, $\gamma$ and $\delta$

$$\alpha \sin (2 + \xi)\, \theta_1 + \beta \sin \xi\theta_1 = 0$$
$$\alpha (2 + \xi) \cos (2 + \xi)\, \theta_1 + \beta\xi \cos \xi\theta_1 = 0$$
$$\gamma \sin (2 - \xi)\, \theta_1 + \delta \sin \xi\theta_1 = 0$$
$$\gamma (2 - \xi) \cos (2 - \xi)\, \theta_1 + \delta\xi \cos \xi\theta_1 = 0$$

It can be verified that for $\xi \neq 0$ and $0 < \theta_1 \leqslant \pi$ this system of equations has no nonzero solutions for constant $\alpha$, $\beta$, $\gamma$, $\delta$. Consequently, for the optimal body the exponent $n$ cannot be larger than two, i.e. we have the case $n = 2$, $m + n \geqslant 4$. The function $f$ then has the form $f = \alpha \sin 2\theta + \beta\theta$. Substituting this expression into the boundary conditions for the function $\psi$, we obtain

$$\alpha \sin 2\theta_1 + \beta\theta_1 = 0, \quad 2\alpha \cos 2\theta_1 + \beta = 0$$

Equating the determinant of this system of equations in $\alpha$ and $\beta$ to zero, we obtain an equation for $\theta_1$, namely, $\operatorname{tg} 2\theta_1 = 2\theta_1$. The root of this equation has the approximate value $\theta_1 \approx 128.7°$. Thus, in the plane-parallel case the optimal body has an angle at the stagnation point equal to $2(\pi - \theta_1) \approx 102.6°$.

**5.** We now consider axially symmetric flow. We introduce a cylindrical system of coordinates $r$, $z$. The functions $\mathbf{v}$ and $\mathbf{v}^*$ then admit stream functions $\psi$ and $\psi^*$, which are definable by the equations. In this case, from Eqs. (1.1) and (2.6) we can obtain equations for the functions $\psi$ and $\psi^*$

$$K^2\psi = rR\left[\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\left(\frac{1}{r^2}K\psi\right) - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{r^2}K\psi\right)\right]$$
$$K^2\psi^* - 2K^2\psi = rR\left[\left(\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\right)\left(\frac{1}{r^2}K\psi^*\right) - \right.$$
$$\left. 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi^*}{\partial z}\right)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) + 2\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi^*}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{r}\right)\right]$$
$$K = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The boundary conditions for the stream functions and the optimality condition assume, in this case, the form

$$\psi|_S = \frac{\partial\psi}{\partial n}\bigg|_S = 0, \quad \frac{\partial\psi}{\partial z}\bigg|_\Sigma = ru_r, \quad \frac{\partial\psi}{\partial r}\bigg)\bigg|_\Sigma = -ru_z$$
$$\psi^*|_S = \frac{\partial\psi^*}{\partial n}\bigg|_S = \psi^*|_\Sigma = \frac{\partial\psi^*}{\partial n}\bigg|_\Sigma = 0, \quad \frac{1}{r^2}\frac{\partial^2\psi}{\partial n^2}\left(\frac{\partial^2\psi}{\partial n^2} - \frac{\partial^2\psi^*}{\partial n^2}\right)\bigg|_S = \text{const}$$

We can carry through, in the axially symmetric case, just as we did in the plane-parallel case, an investigation of the asymptotic behavior of the functions $\psi$ and $\psi^*$ close to the stagnation point. We give the final result, omitting the intermediate calculations

$$\psi = a\rho^3(\cos\theta + {}^1/_2)(\cos\theta - 1)^2 + o(\rho^3) \tag{5.1}$$
$$\psi^* = b\rho^3(\cos\theta + {}^1/_2)(\cos\theta - 1)^2 + o(\rho^3)$$

where $a$ and $b$ are arbitrary constants, $\rho$ is the distance to the stagnation point, and $\theta$ is the angle reckoned from the axis of symmetry. It follows from the expressions (5.1) that the optimal body has, close to the stagnation point, a conical shape with apex angle of 120°.

We note that the terms discarded in Eqs. (4.6) are of the order $\rho^2 R$ and, therefore, in satisfying the inequality $\rho^2 R \ll 1$ ( $\rho^3 R \ll 1$ in the axially symmetric case) we can, with a high degree of accuracy, assume the flow to be Stokes flow. Therefore, in the plane-parallel case, and also in the axially symmetric case, the magnitude of the angle $\theta_1$ depends neither on the Reynolds number nor on whether the singular point is at the front or at the back.

In conclusion, the author thanks F. L. Chernous'ko for his statement of the problem and N. V. Banichuk for useful discussions.

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# PROBLEM WITH DISCONTINUOUS BOUNDARY CONDITIONS AND THE DIFFUSION BOUNDARY LAYER APPROXIMATION

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We consider the stationary convective diffusion problem (heat conductivity problem) which occurs in the flow of a fluid with a shear velocity profile above an infinite plate. On the plate we assume discontinuous boundary conditions of zero flow, zero concentration type. This problem is solved by use of the Wiener-Hopf method with longitudinal diffusion taken into account. We obtain the exact solution in the form of a complex integral and we determine an asymptotic expansion for the density of the flow on the plate close to and far from a discontinuity point in the boundary conditions. We show that close to this point the diffusion boundary layer approximation (DBLA) is unsuitable. We determine the character of the singularity in the flow density at the discontinuity point and we make corrections to the DBLA.

**1. Statement of the problem and the Wiener-Hopf method.** The mathematical statement of our problem is the following:

$$2Vy\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}, \quad 0 < x < \infty, \quad y > 0, \quad V > 0 \tag{1.1}$$

$$\frac{\partial C}{\partial y}(x, 0) = 0, \quad x < 0; \quad C(x, 0) = 0, \quad x > 0 \tag{1.2}$$

$$C(x, y) \to 1, \quad x \to -\infty \quad \text{or} \quad y \to \infty$$

We seek a bounded solution of this problem. All variables are assumed to be dimensionless.